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Convergence of line-renormalized expansions in turbulence theory

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Passive scalar convection by a prescribed random velocity field is represented in terms of integral equations. Primitive perturbation expansions are constructed by iterating these integral equation representations as in Kraichnan (1977). First and second iterations of elemental functions within these expansions are assumed quadratically integrable with respect to space and time. That is, they are assumed to belong to the space L_2. Line-renormalized perturbation expansions are constructed, corresponding to these primitive perturbation expansions, which converge almost everywhere. The direct-interaction approximation and the Lagrangian-history direct-interaction approximation are the simplest truncations of the appropriate line-renormalized perturbation expansions.

1. Introduction

Kraichnan (1977) has presented a systematic construction of Eulerian and Lagrangian-history renormalized expansions in turbulence theory. Corresponding primitive perturbation expansions are similar to solutions of the Volterra integral equation of the second kind as presented in Tricomi (1957):

$$\phi(x) = f(x) + \lambda \int_0^x K(x,y)\,\phi(y)\,dy \quad (0 \leqslant x \leqslant h). \tag{1.1}$$

When the kernel $K(x,y)$ and the function $f(x)$ belong to the class L_2, there is one and essentially one solution in the same class L_2. Functions non-zero on only a set of measure zero are ignored. This solution is given by

$$\phi(x) = f(x) - \lambda \int_0^x H(x,y:\lambda) f(y)\,dy, \tag{1.2}$$

where the 'resolvent kernel' is

$$-H(x,y:\lambda) = \sum_{v=0}^{\infty} \lambda^\nu K_{\nu+1}(x,y),$$

and the series converges almost everywhere. The iterated kernels are defined as

$$K_{\nu+1}(x,y) = \int_0^x (K(x,z)\,K_\nu(z,y)\,dz \quad (\nu = 1,2,3,\ldots).$$

Letting $K(x,y) = f(y)\,k(x,y)$ implies that

$$f(x) = \phi(x) - \lambda \int_0^x \phi(y)\,k(x,y) f(y)\,dy \quad (0 \leqslant x \leqslant h),$$

or in the form of the solution to (1.1),

$$f(x) = \phi(x) - \lambda \int_0^x h(x, y : \lambda)\, \phi(y)\, dy, \tag{1.3}$$

where

$$-h(x, y : \lambda) = \sum_{v=0}^{\infty} \lambda^{\nu} K_{\nu+1}(x, y),$$

$$K_{\nu+1}(x, y) = -\int_0^x \phi(z)\, k(x, z)\, K_{\nu}(z, y)\, dz \quad (\nu = 1, 2, 3, \ldots).$$

This is an expansion of $f(x)$ in terms of $\phi(x)$, a reversion of the expansion in (1.2) which converges almost everywhere.

We construct primitive perturbation expansions as in Kraichnan (1977), and analogous to (1.2). Reversion expansions are generated as in (1.3), our modification of the work of Kraichnan (1977). Convergence properties of the consequent renormalized expansions follow from those of the primitive perturbation expansions and their reversions.

2. Eulerian renormalization

We consider passive scalar dynamics as defined by

$$\left\{\frac{\partial}{\partial t} - \kappa \nabla^2\right\} \psi(\mathbf{x}, t) = -\mathbf{u}(\mathbf{x}, t) . \nabla \psi(\mathbf{x}, t), \quad \nabla . \mathbf{u}(\mathbf{x}, t) = 0, \tag{2.1}$$

where κ is a constant diffusivity and $\mathbf{u}(\mathbf{x}, t)$, a prescribed velocity field whose statistics are assumed isotropic and stationary. The Green function corresponding to the scalar field $\psi(\mathbf{x}, t)$ obeys equations

$$\left\{\frac{\partial}{\partial t} - \kappa \nabla^2\right\} \hat{G}(\mathbf{x}, t : \mathbf{x}', t') = -u_i(\mathbf{x}, t) \frac{\partial}{\partial x_i} \hat{G}(\mathbf{x}, t : \mathbf{x}', t') \quad (t \geqslant t'), \tag{2.2}$$

$$\hat{G}(\mathbf{x}, t : \mathbf{x}', t) = \delta(\mathbf{x} - \mathbf{x}'), \tag{2.3}$$

summing repeated indices $i = 1, 2, 3$. This is equivalent to the integral equation

$$\hat{G}(\mathbf{x}, t : \mathbf{x}', t') = G^0(\mathbf{x}, t : \mathbf{x}', t') - \int_{t'}^{t} ds \int d\mathbf{y}\, G^0(\mathbf{x}, t : \mathbf{y}, s) \left\{u_k(\mathbf{y}, s) \frac{\partial}{\partial y_k} \hat{G}(\mathbf{y}, s : \mathbf{x}' t')\right\}, \tag{2.4}$$

where the spatial integral is definite extending over the spatial domain, and $G^0(\mathbf{x}, t : \mathbf{x}', t')$ satisfies

$$\left\{\frac{\partial}{\partial t} - \kappa \nabla^2\right\} G^0(\mathbf{x}, t : \mathbf{x}' t') = 0.$$

Equation (2.4) is very similar to a Volterra integral equation of the second kind, (1.1).

We ensemble-average (2.2), obtaining

$$\left\{\frac{\partial}{\partial t} - \kappa \nabla^2\right\} G(\mathbf{x}, t : \mathbf{x}', t') = -\left\langle u_i(\mathbf{x}, t) \frac{\partial}{\partial x_i} \hat{G}(\mathbf{x}, t : \mathbf{x}', t') \right\rangle, \tag{2.5}$$

where $G(\mathbf{x}, t : \mathbf{x}', t') = \langle \hat{G}(\mathbf{x}, t : \mathbf{x}', t') \rangle$. The aim of Eulerian renormalization is to represent $\langle u_i(\mathbf{x}, t)\, (\partial/\partial x_i)\, \hat{G}(\mathbf{x}, t : \mathbf{x}', t') \rangle$ as an expansion in terms of $G(\mathbf{x}, t : \mathbf{x}', t')$, and thus be able to solve (2.5).

The iteration solution of (2.4) has the form

$$u_i(\mathbf{x},t)\frac{\partial}{\partial x_i}\hat{G}(\mathbf{x},t:\mathbf{x}',t')$$
$$= u_i(\mathbf{x},t)\frac{\partial}{\partial x_i}G^0(\mathbf{x},t:\mathbf{x}',t') - \int_{t'}^{t} ds \int d\mathbf{y}\left\{u_i(\mathbf{x},t)\frac{\partial}{\partial x_i}G^0(\mathbf{x},t:\mathbf{y},s)\right\}$$
$$\times\left\{u_k(\mathbf{y},s)\frac{\partial}{\partial y_k}G^0(\mathbf{y},s:\mathbf{x}',t')\right\} + \dots. \tag{2.6}$$

Defining the 'iterated kernel',

$$u_i(\mathbf{x},t)\frac{\partial}{\partial x_i}G^0_n(\mathbf{x},t:\mathbf{x}',t') = \int_{t'}^{t} ds \int d\mathbf{y}\left\{u_i(\mathbf{x},t)\frac{\partial}{\partial x_i}G^0(\mathbf{x},t:\mathbf{y},s)\right\}$$
$$\times\left\{u_k(\mathbf{y},s)\frac{\partial}{\partial y_k}G^0_{n-1}(\mathbf{y},s:\mathbf{x}',t')\right\},$$

or equivalently

$$u_i(\mathbf{x},t)\frac{\partial}{\partial x_i}G^0_n(\mathbf{x},t:\mathbf{x}',t')$$
$$= \int_{t'}^{t} ds \int d\mathbf{y}\left\{u_i(\mathbf{x},t)\frac{\partial}{\partial x_i}G^0_j(\mathbf{x},t:\mathbf{y},s)\right\}\left\{u_k(\mathbf{y},s)\frac{\partial}{\partial y_k}G^0_{n-1-j}(\mathbf{y},s:\mathbf{x}',t')\right\}, \tag{2.7}$$

where j is any of $0, 1, \dots, n-1$, and

$$u_i(\mathbf{x},t)\frac{\partial}{\partial x_i}G^0_0(\mathbf{x},t:\mathbf{x}',t') \equiv u_i(\mathbf{x},t)\frac{\partial}{\partial x_i}G^0(\mathbf{x},t:\mathbf{x}',t'),$$

the iteration solution may be written as

$$u_i(\mathbf{x},t)\frac{\partial}{\partial x_i}\hat{G}(\mathbf{x},t:\mathbf{x}',t') = \sum_{j=0}^{\infty}(-1)^j u_i(\mathbf{x},t)\frac{\partial}{\partial x_i}G^0_j(\mathbf{x},t:\mathbf{x}',t'). \tag{2.8}$$

The ensemble average of this is the primitive perturbation expansion,

$$\left\langle u_i(\mathbf{x},t)\frac{\partial}{\partial x_i}\hat{G}(\mathbf{x},t:\mathbf{x}',t')\right\rangle = \sum_{j=0}^{\infty}(-1)^j\left\langle u_i(\mathbf{x},t)\frac{\partial}{\partial x_i}G^0_j(\mathbf{x},t:\mathbf{x}',t')\right\rangle. \tag{2.9}$$

We seek upper bounds for the terms of each order of this expansion. Given any functions $\zeta(x)$ and $\eta(x)$ in the L_2 space, the norm is

$$\|\zeta\|^2 = \int_0^h \zeta^2(x)\,dx < \infty,$$

and the Schwartz inequality yields

$$\left\{\int_a^b \zeta(x)\,\eta(x)\,dx\right\}^2 \leqslant \int_a^b \zeta(x)^2\,dx \int_a^b \eta(x)^2\,dx, \tag{2.10}$$

where $[a,b]$ may be within $[0,h]$. Assuming $G^0_l(\mathbf{x},t:\mathbf{x}',t')$ and $u_i(\mathbf{x},t)\,(\partial/\partial x_i)\,G^0_l(\mathbf{x},t:\mathbf{x}',t')$ are in the space L_2 where $l = 1, 2$ from appendix 1, then

$$\left\{u_i(\mathbf{x},t)\frac{\partial}{\partial x_i}G^0_3(\mathbf{x},t:\mathbf{x}',t')\right\}^2$$
$$\leqslant \int ds\int d\mathbf{y}\left\{u_i(\mathbf{x},t)\frac{\partial}{\partial x_i}G^0_1(\mathbf{x},t:\mathbf{y},s)\right\}^2\int ds\int d\mathbf{y}\left\{u_k(\mathbf{y},s)\frac{\partial}{\partial y_k}G^0_1(\mathbf{y},s:\mathbf{x}',t')\right\}^2$$
$$\leqslant f(\mathbf{x},t)\,g(\mathbf{x}',t'),$$

setting

$$f(\mathbf{x},t) = \int ds \int d\mathbf{y} \left\{ u_i(\mathbf{x},t) \frac{\partial}{\partial x_i} G_1^0(\mathbf{x},t:\mathbf{y},s) \right\}^2,$$

$$g(\mathbf{x}',t') = \int ds \int d\mathbf{y} \int u_k(\mathbf{y},s) \frac{\partial}{\partial y_k} G_1^0(\mathbf{y},s:\mathbf{x}',t') \Big\}^2.$$

Time integrals are definite, extending over the time domain. Consequently, (2.7) and (2.10) imply

$$\left\{ u_i(\mathbf{x},t) \frac{\partial}{\partial x_i} G_5^0(\mathbf{x},t:\mathbf{x}',t') \right\}^2 \leqslant f(\mathbf{x},t) \int_{t'}^{t} ds \int d\mathbf{y} \left\{ u_k(\mathbf{y},s) \frac{\partial}{\partial y_k} G_3^0(\mathbf{y},s:\mathbf{x}',t') \right\}^2$$
$$\leqslant f(\mathbf{x},t)\, g(\mathbf{x}',t')\{F(t)-F(t')\},$$

where

$$\{F(t)-F(t')\} = \int_{t'}^{t} ds \int d\mathbf{y} f(\mathbf{y},s).$$

By induction, this may be extended to

$$\left\{ u_i(\mathbf{x},t) \frac{\partial}{\partial x_i} G_{2n+1}^0(\mathbf{x},t:\mathbf{x}',t') \right\}^2 \leqslant f(\mathbf{x},t)\, g(\mathbf{x}',t') \frac{\{F(t)-F(t')\}^{n-1}}{(n-1)!} \quad (n = 1,2,3,\ldots),$$

and in the same manner

$$\left\{ u_i(\mathbf{x},t) \frac{\partial}{\partial x_i} G_{2n}^0(\mathbf{x},t:\mathbf{x}',t') \right\}^2 \leqslant f(\mathbf{x},t)\, h(\mathbf{x}',t') \frac{\{F(t)-F(t')\}^{n-2}}{(n-2)!} \quad (n = 2,3,4,\ldots),$$

where

$$h(\mathbf{x}',t') = \int ds \int d\mathbf{y} \left\{ u_k(y,s) \frac{\partial}{\partial y_k} G_2^0(\mathbf{y},s:\mathbf{x}',t') \right\}^2.$$

This establishes that the expansion in (2.8) is uniformly convergent almost everywhere.

As in (1.3), our reversion of the primitive perturbation expansion of (2.9) is by considering the iterative solution to (2.4) cast as

$$\frac{\partial}{\partial x_i} G^0(\mathbf{x},t:\mathbf{x}',t') = \frac{\partial}{\partial x_i} \hat{G}(\mathbf{x},t:\mathbf{x}',t') + \int_{t'}^{t} ds \int d\mathbf{y} \left\{ u_k(\mathbf{y},s) \frac{\partial}{\partial y_k} \hat{G}(\mathbf{y},s:\mathbf{x}',t') \right\} \frac{\partial}{\partial x_i} G^0(\mathbf{x},t:\mathbf{y},s).$$

It follows that

$$\frac{\partial}{\partial x_i} G^0(\mathbf{x},t:\mathbf{x}',t') = \sum_{n=0}^{\infty} \left\langle \frac{\partial}{\partial x_i} \hat{G}_n(\mathbf{x},t:\mathbf{x}',t') \right\rangle, \tag{2.11}$$

where $\hat{G}_0(\mathbf{x},t:\mathbf{x}',t') \equiv \hat{G}(\mathbf{x},t:\mathbf{x}',t')$ and

$$\frac{\partial}{\partial x_i} \hat{G}_n(\mathbf{x},t:\mathbf{x}',t') = \int_{t'}^{t} ds \int d\mathbf{y} \left\{ u_k(y,s) \frac{\partial}{\partial y_k} \hat{G}_l(\mathbf{y},s:\mathbf{x}',t') \right\} \frac{\partial}{\partial x_i} \hat{G}_{n-1-l}(\mathbf{x},t:\mathbf{y},s).$$

Upper bounds for terms in the expansion of (2.11) may be obtained by setting

$$\alpha(\mathbf{x}',t') = \int ds \int d\mathbf{y} \left\{ u_k(\mathbf{y},s) \frac{\partial}{\partial y_k} \hat{G}_1(\mathbf{y},s:\mathbf{x}',t') \right\}^2,$$

$$\beta(\mathbf{x},t) = \int ds \int d\mathbf{y} \left\{ \frac{\partial}{\partial x_i} \hat{G}_1(\mathbf{x},t:\mathbf{y},s) \right\}^2,$$

$$\gamma(\mathbf{x},t) = \int ds \int d\mathbf{y} \left\{\frac{\partial}{\partial x_i} \hat{G}_2(\mathbf{x},t:\mathbf{y},s)\right\}^2,$$

$$A(t)-A(t') = \int_{t'}^{t} ds \int d\mathbf{y}\, \alpha(\mathbf{y},s),$$

which imply

$$\left\{\frac{\partial}{\partial x_i} \hat{G}_{2n+1}(\mathbf{x},t:\mathbf{x}',t')\right\}^2 \leqslant \alpha(\mathbf{x}',t')\,\beta(\mathbf{x},t)\frac{\{A(t)-A(t')\}^{n-1}}{(n-1)!} \quad (n = 1,2,3,\ldots),$$

$$\left\{\frac{\partial}{\partial x_i} \hat{G}_{2n}(\mathbf{x},t:\mathbf{x}',t')\right\} \leqslant \alpha(\mathbf{x}',t')\,\gamma(\mathbf{x},t)\,\frac{\{A(t)-A(t')\}^{n-2}}{(n-2)!} \quad (n = 2,3,4,\ldots).$$

Therefore the expansion converges uniformly almost everywhere. We conclude that

$$-\left\langle u_i(\mathbf{x},t)\frac{\partial}{\partial x_i}\hat{G}(\mathbf{x},t:\mathbf{x}',t')\right\rangle$$
$$= \int_{t'}^{t} ds \int d\mathbf{y} \langle u_i(\mathbf{x},t)\,u_k(\mathbf{y},s)\rangle \frac{\partial}{\partial x_i} G(\mathbf{x},t:\mathbf{y},s)\frac{\partial}{\partial y_k} G(\mathbf{y},s:\mathbf{x}',t') + \ldots, \quad (2.12)$$

where (2.11) has been substituted into (2.9) and terms have been rearranged. This is a renormalized perturbation expansion uniformly convergent almost everywhere. Neglecting all but the first term is the direct-interaction approximation.

Returning to (2.1) for the scalar field $\psi(\mathbf{x},t)$, we may posit the equivalent integral equation

$$\psi(\mathbf{x},t) = \psi^0(\mathbf{x},t) - \int_{t_0}^{t} ds \int d\mathbf{y}\; G^0(\mathbf{x},t:\mathbf{y},s)\left\{u_k(\mathbf{y},s)\frac{\partial}{\partial y_k}\psi(\mathbf{y},s)\right\}, \quad (2.13)$$

where

$$\left\{\frac{\partial}{\partial t} - \kappa\nabla^2\right\}\psi^0(\mathbf{x},t) = 0.$$

The covariance satisfies

$$\left\{\frac{\partial}{\partial t} - \kappa\nabla^2\right\}\langle\psi(\mathbf{x},t)\,\psi(\mathbf{x}',t')\rangle = -\left\langle u_i(\mathbf{x},t)\frac{\partial}{\partial x_i}\psi(\mathbf{x},t)\,\psi(\mathbf{x}',t')\right\rangle,$$

which is solvable when an alternative expression for $\langle u_i(\mathbf{x},t)\,(\partial/\partial x_i)\,\psi(\mathbf{x},t)\,\psi(X',t')\rangle$ in terms of $\langle\psi(\mathbf{x},t)\,\psi(\mathbf{x}',t')\rangle$ is found.

We proceed from (2.13) as in the analysis of (2.4), and establish the expansion,

$$\psi(\mathbf{x},t) = \sum_{j=0}^{\infty} (-1)^j \psi_j^0(\mathbf{x},t), \quad \psi_0^0(\mathbf{x},t) \equiv \psi^0(\mathbf{x},t),$$

where

$$\psi_n^0(\mathbf{x},t) = \int_{t_0}^{t} ds \int d\mathbf{y}\, G_j^0(\mathbf{x},t:\mathbf{y},s)\left\{u_k(\mathbf{y},s)\frac{\partial}{\partial y_k}\psi_{n-1-j}^0(\mathbf{y},s)\right\},$$

and j is any of $0, 1, \ldots, n-1$. Its reversion is

$$\psi^0(\mathbf{x},t) = \sum_{j=0}^{\infty} \psi_j(\mathbf{x},t), \quad \psi_0(\mathbf{x},t) \equiv \psi(\mathbf{x},t),$$

via the iteration solution to

$$\psi^0(\mathbf{x},t) = \psi(\mathbf{x},t) + \int_{t_0}^{t} ds \int d\mathbf{y} \left\{u_k(\mathbf{y},s)\frac{\partial}{\partial y_k}\psi(\mathbf{y},s)\right\} G^0(\mathbf{x},t:\mathbf{y},s),$$

where, analogously,

$$\psi_n(\mathbf{x},t) = \int_{t_0}^{t} ds \int d\mathbf{y} \left\{ u_k(\mathbf{y},s) \frac{\partial}{\partial y_k} \psi_j(\mathbf{y},s) \right\} \hat{G}_{n-1-j}(\mathbf{x},t:\mathbf{y},s).$$

Assembling these, it follows that

$$-\left\langle u_i(\mathbf{x},t) \frac{\partial}{\partial x_i} \psi(\mathbf{x},t)\, \psi(\mathbf{x}',t') \right\rangle$$
$$= \int_{t_0}^{t} ds \int d\mathbf{y} \, \langle u_i(\mathbf{x},t)\, u_k(\mathbf{y},s) \rangle \frac{\partial}{\partial x_i} G(\mathbf{x},t:\mathbf{y},s) \frac{\partial}{\partial y_k} \langle \psi(\mathbf{y},s)\, \psi(\mathbf{x}',t') \rangle$$
$$+ \int_{t_0}^{t'} ds \int d\mathbf{y} \, \langle u_i(\mathbf{x},t)\, u_k(\mathbf{y},s) \rangle\, G(\mathbf{x}',t':\mathbf{y},s) \frac{\partial^2}{\partial x_i\, \partial y_k} \langle \psi(\mathbf{x},t)\, \psi(\mathbf{y},s) \rangle + \ldots,$$

which is a renormalized perturbation expansion uniformly convergent almost everywhere. The direct-interaction approximation is obtained when only the first two terms are retained.

3. Lagrangian renormalization

We place the dynamics of the passive scalar field in a quasi-Lagrangian mode of description, writing the governing equations as

$$\left\{ \frac{\partial}{\partial t} - \kappa \nabla^2 \right\} \Psi(\mathbf{x},t|t) = -u_i(\mathbf{x},t|t) \frac{\partial}{\partial x_i} \Psi(\mathbf{x},t|t), \tag{3.1}$$

$$\frac{\partial}{\partial x_i} u_i(\mathbf{x},t|t) = 0, \tag{3.2}$$

$$\frac{\partial}{\partial t} \Psi(\mathbf{x},t|s) = -u_i(\mathbf{x},t|t) \frac{\partial}{\partial x_i} \Psi(\mathbf{x},t|s), \tag{3.3}$$

where, as in Kraichnan (1977), $u(\mathbf{x},t|s)$ is the generalized velocity, and $\Psi(\mathbf{x},\ t|s)$ the generalized passive scalar field. That is, if a material element passes through space–time co-ordinates $(\mathbf{x},t)$ and is observed at time s, it will have velocity $\mathbf{u}(\mathbf{x},\ t|s)$ and passive scalar field value $\Psi(\mathbf{x},t|s)$. Equation (3.1) states that, when labelling time t and measuring time s are the same, the quasi-Lagrangian picture becomes the Eulerian picture. Equation (3.2) is the incompressibility constraint. Equation (3.3) is a kinematic constraint on quasi-Lagrangian field variables. In following the flow

$$\Psi(\mathbf{x},t|s) = \Psi(\mathbf{x}+\delta\mathbf{x},\ t+\delta t|s),$$

which may be Taylor-expanded to

$$\Psi(\mathbf{x},t|s) \simeq \Psi(\mathbf{x},t|s) + \delta\mathbf{x} \,.\, \nabla\Psi(\mathbf{x},t|s) + \delta t \frac{\partial}{\partial t} \Psi(\mathbf{x},t|s).$$

The limit of small δt is (3.3). As in the discussion in §2, κ is a constant diffusivity and $\mathbf{u}(\mathbf{x},t|s)$ a prescribed velocity field having isotropic stationary statistics. The correspon-ponding Green function satisfies equations

$$\left\{ \frac{\partial}{\partial t} - \kappa \nabla^2 \right\} \hat{G}(\mathbf{x},t|t:\mathbf{x}',t'|s') = -u_i(\mathbf{x},t|t) \frac{\partial}{\partial x_i} \hat{G}(\mathbf{x},t|t:\mathbf{x}',t'|s'), \tag{3.4}$$

$$\frac{\partial}{\partial t} \hat{G}(\mathbf{x},t|s:\mathbf{x}',t'|s') = -u_i(\mathbf{x},t|t) \frac{\partial}{\partial x_i} \hat{G}(\mathbf{x},t|s:\mathbf{x}',t'|s'), \tag{3.5}$$

$$\hat{G}(\mathbf{x},t|s:\mathbf{x}',t|s) = \delta(\mathbf{x}-\mathbf{x}'),$$

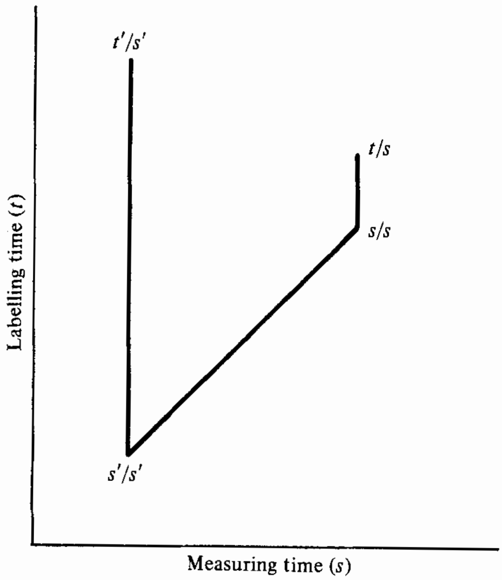

FIGURE 1. Integration path for (3.6). s is always greater than or equal to s', whereas t and t' are unrestricted.

which are equivalent to the integral equation,

$$\hat{G}(\mathbf{x},t|s:\mathbf{x}',t'|s') = G^0(\mathbf{x},t|s:\mathbf{x}',t'|s') + \int_{t'}^{s'} dr \int d\mathbf{y}\, G^0(\mathbf{x},t|s:\mathbf{y},r|s')\left\{-u_k(\mathbf{y},r|r)\frac{\partial}{\partial y_k}\hat{G}(\mathbf{y},r|s':\mathbf{x}',t'|s')\right\} + \int_{s'}^{s} dr \int d\mathbf{y}\, G^0(\mathbf{x},t|s:\mathbf{y},r|r)\left\{-u_k(\mathbf{y},r|r)\frac{\partial}{\partial y_k}\hat{G}(\mathbf{y},r|r:\mathbf{x}',t'|s')\right\} + \int_{s}^{t} dr \int d\mathbf{y}\, G^0(\mathbf{x},t|s:\mathbf{y},r|s)\left\{-u_k(\mathbf{y},r|r)\frac{\partial}{\partial y_k}\hat{G}(\mathbf{y},r|s:\mathbf{x}',t'|s')\right\}, \tag{3.6}$$

where spatial integrals are definite, extending over the spatial domain, and

$$G^0(\mathbf{x},t|s:\mathbf{x}',t'|s')$$

obeys

$$\left\{\frac{\partial}{\partial t}-\kappa\nabla^2\right\} G^0(\mathbf{x},t|t:\mathbf{x}',t'|s') = 0,$$

$$\frac{\partial}{\partial t}G^0(\mathbf{x},t|s:\mathbf{x}',t'|s') = 0,$$

$$\frac{\partial}{\partial t'}G^0(\mathbf{x},t|s:\mathbf{x}',t'|s') = 0.$$

The path integral for $\hat{G}(\mathbf{x},t|s:\mathbf{x}',t'|s')$ is as in figure 1. The ensemble average of (3.4) and (3.5) is

$$\left\{\frac{\partial}{\partial t}-\kappa\nabla^2\right\} G(\mathbf{x},t|t:\mathbf{x}',t'|s') = -\left\langle u_i(\mathbf{x},t|t)\frac{\partial}{\partial x_i}\hat{G}(\mathbf{x},t|t:\mathbf{x}',t'|s')\right\rangle, \tag{3.7}$$

$$\frac{\partial}{\partial t}G(\mathbf{x},t|s:\mathbf{x}',t'|s') = -\left\langle u_i(\mathbf{x},t|t)\frac{\partial}{\partial x_i}\hat{G}(\mathbf{x},t|s:\mathbf{x}',t'|s')\right\rangle, \tag{3.8}$$

where $G(\mathbf{x},t|s:\mathbf{x}',t'|s') = \langle \hat{G}(\mathbf{x},t|s:\mathbf{x}',t'|s')\rangle$. We must represent

$$\left\langle u_i(\mathbf{x},t|t)\frac{\partial}{\partial x_i}\hat{G}(\mathbf{x},t|s:\mathbf{x}',t'|s')\right\rangle$$

as an expansion in terms of $G(\mathbf{x},t|s:\mathbf{x}',t'|s')$ to solve (3.7) and (3.8).

The iteration solution to (3.6) is

$$\hat{G}(\mathbf{x},t|s:\mathbf{x}',t'|s') = \sum_{j=0}^{\infty}(-1)^j G_j^0(\mathbf{x},t|s:\mathbf{x}',t'|s'), \tag{3.9}$$

where the 'iterated kernel' is defined as

$$\begin{aligned} G_n^0(\mathbf{x},t|s:\mathbf{x}',t'|s') &= \int_{t'}^{s'} dr\int d\mathbf{y}\, G_j^0(\mathbf{x},t|s:\mathbf{y},r|s')\,u_k(\mathbf{y},r|r)\frac{\partial}{\partial y_k}G_{n-1-j}^0(\mathbf{y},r|s':\mathbf{x}',t'|s') \\ &+ \int_{s'}^{s} dr\int d\mathbf{y}\, G_j^0(\mathbf{x},t|s:\mathbf{y},r|r)\,u_k(\mathbf{y},r|r)\frac{\partial}{\partial y_k}G_{n-1-j}^0(\mathbf{y},r|r:\mathbf{x}',t'|s') \\ &+ \int_{s}^{t} dr\int d\mathbf{y}\, G_j^0(\mathbf{x},t|s:\mathbf{y},r|s)\,u_k(\mathbf{y},r|r)\frac{\partial}{\partial y_k}G_{n-1-j}^0(\mathbf{y},r|s:\mathbf{x}',t'|s'), \end{aligned} \tag{3.10}$$

$$G_0^0(\mathbf{x},t|s:\mathbf{x}',t'|s') = G^0(\mathbf{x},t|s:\mathbf{x}',t'|s'),$$

and j may take any of the values $0, 1, 2, \ldots, n-1$.

Assuming $G_l^0(\mathbf{x},t|s:\mathbf{x}',t'|s')$ and $u_i(\mathbf{x},t|t)\,(\partial/\partial x_i)\,G_l^0(\mathbf{x},t|s:\mathbf{x}',t'|s')$ belong to the space L_2, where $l = 1, 2$, upper bounds are sought for the terms of each order of (3.9). The Schwartz inequality implies that, given any functions $\zeta(x,y)$ and $\eta(x,y)$ in the space L_2,

$$\|\zeta\|^2\,\|\eta\|^2 \geqslant \left\|\int_{a_0}^{b_0}\zeta(x,y)\,\eta(z,y)\,dy\right\|^2, \tag{3.11}$$

as pointed out in Tricomi (1957). The interval $[a_0, b_0]$ is a subset of the domain of ζ and η with respect to y. Equation (3.10) and the Schwartz inequality imply

$$\begin{aligned} &\left\{u_i(\mathbf{x},t|t)\frac{\partial}{\partial x_i}G_3^0(\mathbf{x},t|t:\mathbf{x}',t'|t')\right\}^2 \\ &\leqslant \int dr\int d\mathbf{y}\left\{u_i(\mathbf{x},t|t)\frac{\partial}{\partial x_i}G_1^0(\mathbf{x},t|t:\mathbf{y},r|r)\right\}^2\int dr\int d\mathbf{y}\left\{u_k(\mathbf{y},r|r)\frac{\partial}{\partial y_k}G_1^0(\mathbf{y},r|r:\mathbf{x}',t'|t')\right\}^2 \\ &\leqslant f(\mathbf{x},t)\,g(\mathbf{x}',t'), \end{aligned}$$

where

$$f(\mathbf{x},t) = \int dr\int d\mathbf{y}\left\{u_i(\mathbf{x},t|t)\frac{\partial}{\partial x_i}G_1^0(\mathbf{x},t|t:\mathbf{y},r|r)\right\}^2,$$

$$g(\mathbf{x}',t') = \int dr\int d\mathbf{y}\left\{u_k(\mathbf{y},r|r)\frac{\partial}{\partial y_k}G_1^0(\mathbf{y},r|r:\mathbf{x}',t'|t')\right\}^2,$$

and time integrals are definite, extending over the time domain. The generalization that may be induced is

$$\left\{u_i(\mathbf{x},t|t)\frac{\partial}{\partial x_i}G_{2n+1}^0(\mathbf{x},t|t:\mathbf{x}',t'|t')\right\}^2 \leqslant f(\mathbf{x},t)\,g(\mathbf{x}',t')\frac{\{F(t)-F(t')\}^{n-1}}{(n-1)!} \quad (n = 1, 2, \ldots), \tag{3.12}$$

and in the same manner

$$\left\{u_i(\mathbf{x},t|t)\frac{\partial}{\partial x_i}G_{2n}^0(\mathbf{x},t|t:\mathbf{x}',t'|t')\right\}^2 \leqslant f(\mathbf{x},t)\,h(\mathbf{x}',t')\frac{\{F(t)-F(t')\}^{n-2}}{(n-2)!} \quad (n = 2, 3, \ldots),$$

where

$$F(t)-F(t') = \int_{t'}^{t} dr \int d\mathbf{y} f(\mathbf{y}, r),$$

$$h(\mathbf{x}', t') = \int dr \int d\mathbf{y} \left\{ u_k(\mathbf{y}, r|r) \frac{\partial}{\partial y_k} G_2^0(\mathbf{y}, r|r : \mathbf{x}', t'|t') \right\}^2.$$

Similarly it follows that

$$\left\{ u_i(\mathbf{x}, t|t) \frac{\partial}{\partial x_i} G_{2n+1}^0(\mathbf{x}, t|s' : \mathbf{x}', t'|s') \right\}^2 \leqslant \mathscr{f}(\mathbf{x}, t, s') \mathscr{g}(\mathbf{x}', t', s') \frac{\{\mathscr{F}(t, s') - \mathscr{F}(t', s')\}^{n-1}}{(n-1)!} \quad (n = 1, 2, \ldots) \qquad (3.13a)$$

$$\left\{ u_i(\mathbf{x}, t|t) \frac{\partial}{\partial x_i} G_{2n}^0(\mathbf{x}, t|s' : \mathbf{x}', t'|s') \right\}^2 \leqslant \mathscr{f}(\mathbf{x}, t, s') \mathscr{h}(\mathbf{x}', t', s') \frac{\{\mathscr{F}(t, s') - \mathscr{F}(t', s')\}^{n-2}}{(n-2)!} \quad (n = 2, 3, \ldots), \qquad (3.13b)$$

where

$$\mathscr{f}(\mathbf{x}, t, s') = \int dr \int d\mathbf{y} \left\{ u_i(\mathbf{x}, t|t) \frac{\partial}{\partial x_i} G_1^0(\mathbf{x}, t|s' : \mathbf{y}, r|s') \right\}^2,$$

$$\mathscr{g}(\mathbf{x}', t', s') = \int dr \int d\mathbf{y} \left\{ u_k(\mathbf{y}, r|r) \frac{\partial}{\partial y_k} G_1^0(\mathbf{y}, r|s' : \mathbf{x}', t'|s') \right\}^2,$$

$$\mathscr{h}(\mathbf{x}', t', s') = \int dr \int d\mathbf{y} \left\{ u_k(\mathbf{y}, r|r) \frac{\partial}{\partial y_k} G_2^0(\mathbf{y}, r|s' : \mathbf{x}', t'|s') \right\}^2,$$

$$\mathscr{F}(t, s') - \mathscr{F}(t', s') = \int_{t'}^{t} dr \int d\mathbf{y}\, \mathscr{f}(\mathbf{y}, r, s').$$

From the triangle inequality and the inequality of (3.11), (3.10) implies

$$\left\| u_i(\mathbf{x}, t|t) \frac{\partial}{\partial x_i} G_{2n+1}^0(\mathbf{x}, t|t : \mathbf{x}', t'|s') \right\| \leqslant \left\| u_i(\mathbf{x}, t|t) \frac{\partial}{\partial x_i} G_j^0(\mathbf{x}, t|t : \mathbf{y}, r|s') \right\| \left\| u_k(\mathbf{y}, r|r) \frac{\partial}{\partial y_k} G_{2n-j}^0(\mathbf{y}, r|s' : \mathbf{x}', t'|s') \right\| + \left\| u_i(\mathbf{x}, t|t) \frac{\partial}{\partial x_i} G_j^0(\mathbf{x}, t|t : \mathbf{y}, r|r) \right\| \left\| u_k(\mathbf{y}, r|r) \frac{\partial}{\partial y_k} G_{2n-j}^0(\mathbf{y}, r|r : \mathbf{x}', t'|s') \right\|, \qquad (3.14)$$

which becomes, upon substitution of (3.12) and (3.13),

$$\left\| u_i(\mathbf{x}, t|t) \frac{\partial}{\partial x_i} G_{2n+1}^0(\mathbf{x}, t|t : \mathbf{x}, t'|s') \right\| \leqslant \left\| u_i(\mathbf{x}, t|t) \frac{\partial}{\partial x_i} G_j^0(\mathbf{x}, t|t : \mathbf{y}, r|\, s') \right\| \left\| \left\{ \mathscr{f}(\mathbf{y}, r, s')\, \mathscr{g}(\mathbf{x}', t', s') \frac{[\mathscr{F}(r, s') - \mathscr{F}(t', s')]^{2n-1-j}}{(2n-1-j)!} \right\}^{\frac{1}{2}} \right\| + \left\| \left\{ f(\mathbf{x}, t)\, g(\mathbf{y}, r) \frac{[F(t) - F(r)]^{j-1}}{(j-1)!} \right\}^{\frac{1}{2}} \right\| \left\| u_k(\mathbf{y}, r|r) \frac{\partial}{\partial y_k} G_{2n-j}^0(\mathbf{y}, r|r : \mathbf{x}', t'|s') \right\|.$$

The assumption

$$\left\| u_i(\mathbf{x}, t|t) \frac{\partial}{\partial x_i} G_l^0(\mathbf{x}, t|s : \mathbf{x}', t'|s') \right\| < N \quad (l = 1, 2)$$

implies

$$\left\| u_i(\mathbf{x}, t|t) \frac{\partial}{\partial x_i} G^0_{2n}(\mathbf{x}, t|t : \mathbf{x}', t'|s') \right\| \leqslant 2^{n-1} N^n,$$

$$\left\| u_i(\mathbf{x}, t|t) \frac{\partial}{\partial x_i} G^0_{2n+1}(\mathbf{x}, t|t : \mathbf{x}', t'|s') \right\| \leqslant 2^n N^{n+1}.$$

When we choose $j = n$ and assume n is even it follows that

$$\begin{aligned} \left\| u_i(\mathbf{x}, t|t) \frac{\partial}{\partial x_i} G^0_{2n+1}(\mathbf{x}, t|t : \mathbf{x}', t'|s') \right\| & \\ \leqslant \frac{2^{\frac{1}{2}n-1} N^{\frac{1}{2}n}}{[(n-1)\,!]^{\frac{1}{2}}} & \left\| \{ \not{f}(\mathbf{y}, r, s')\, \not{g}(\mathbf{x}', t', s')\, [\mathscr{F}(r, s') - \mathscr{F}(t', s')]^{n-1} \}^{\frac{1}{2}} \right\| \\ + \frac{2^{\frac{1}{2}n-1} N^{\frac{1}{2}n}}{[(n-1)\,!]^{\frac{1}{2}}} & \left\| \{ f(\mathbf{x}, t)\, g(\mathbf{y}, r)\, [F(t) - F(r)]^{n-1} \}^{\frac{1}{2}} \right\|. \end{aligned} \tag{3.15}$$

The analysis of $\|u_i(\mathbf{x}, t|t)\,(\partial/\partial x_i)\, G^0_n(\mathbf{x}, t|s : \mathbf{x}', t'|s')\|$ and other expressions is similar. As a consequence of the factorials in the denominators of, for example (3.15), the expansion in (3.9) is pointwise-convergent almost everywhere. Further detail is in appendix 2.

4. Concluding remarks

The discussion centres on the similarity which exists between the Volterra integral equation of the second kind and the integral-equation representations of the passive scalar field and the Green function. Primitive perturbation expansions involve iteration solutions to these integral-equation representations. Corresponding reversion expansions involve iteration solutions of these same integral-equation representations in rearranged form. Given that elemental functions of the primitive perturbation expansions iterated once or twice belong to the L_2 space, the primitive, reversion and line-renormalized perturbation expansions converge almost everywhere. These results and the above-mentioned similarity depend on the presence of molecular diffusivity.

The lowest-order truncation of the renormalized perturbation expansions can be associated with the random-coupling model of Kraichnan (1961). As this is a realizable model, the Green function has a physically acceptable behaviour. Approximations attained by higher-order truncations may not be useful, because this kind of association may not be possible.

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Appendix 1

From (2.7) it may be inferred that

$$\lim_{t\to t'} G_1^0 \propto -\mathbf{u}(\mathbf{x},t)\,.\,(\mathbf{x}-\mathbf{x}')\,\delta(\mathbf{x}-\mathbf{x}'),$$

$$\lim_{t\to t'} \mathbf{u}\,.\,\nabla G_1 \propto -\mathbf{u}(\mathbf{x},t)\,.\,\nabla\{\mathbf{u}(\mathbf{x},t)\,.\,(\mathbf{x}-\mathbf{x}')\,\delta(\mathbf{x}-\mathbf{x}')\},$$

and similarly for G_2^0 and $\mathbf{u}\,.\,\nabla G_2^0$. Integration by parts leads to square-integrability.

Appendix 2

Suppose that functions $\zeta(x)$ and $\eta(x)$ are bounded and measurable, as the elemental functions f, g, $\mathscr{F}$, ... of (3.15) are assumed to be. Then, for any interval $[a,b]$, it follows that

$$\left|\int_a^b dx\,\zeta(x)\,\eta(x)\right| \leqslant |b-a|\,(\sup_{a\leqslant x\leqslant b}|\zeta(x)\,\eta(x)|).$$

The ratio test implies convergence.

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